Triangles in Various Geometries

By Benjamin Stokes and Sebastian Cuervo

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1 Introduction

One of the through lines of high school geometry is the exploration of triangles. We have rigorously examined triangles in Euclidean geometry— but why do we need to specify "Euclidean"? Isn't there just the one geometry we have come to know and love? As it turns out, there are numerous types of geometry, some of which are easier to visualize than others. This paper aims to explore the intriguing nature of triangles (and constructions derived from them) in spherical and hyperbolic geometries. We begin by introducing spherical geometry and the concept of axiomatic systems, then expand into the hyperbolic plane via the examination its foundational axioms. We finish with an exploration of similarly rigorous axioms in single elliptic geometry.

2 Spherical Geometry

Spherical geometry is a branch of mathematics applied in multiple fields including astronomy and global navigation. For example, navigators use the sphere to traverse the globe in the most direct and efficient ways possible while astronomers can calculate the distance from our globe to other planets and stars.

Unlike a straight line in Euclidean geometry, the shortest path on the surface of the sphere connecting two different points is a great circle, such as the equator or a circle with the largest circumference of a sphere.

Definition 2.1. Circles on the sphere centered at the origin are called **great circles**.

Great circles in spherical geometry are used much as straight lines are used in Euclidean space. Unlike how lines in Euclidean space intersect in at most one point, great circles always intersect at two opposite points called *antipodal points.*

In Euclidean geometry, for every line k and a point P not on k, there exists a parallel lines through P that does not intersect the line. However, clearly *something* is different about lines on spheres— great circles always intersect, so in this way there are no parallel lines on the sphere.

Definition 2.2. The regions created by the intersection of two great circles are called *lunes*.

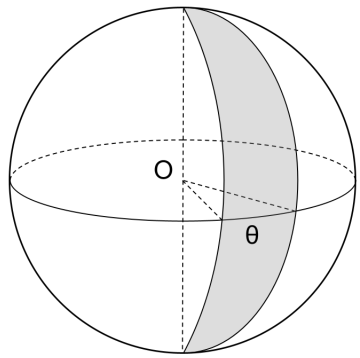

Figure 1: Lune on a Sphere of angle θ.

When a third great circle with no common point of intersection to the other lines is created, the four previous lunes become eight spherical triangles like in the picture below.

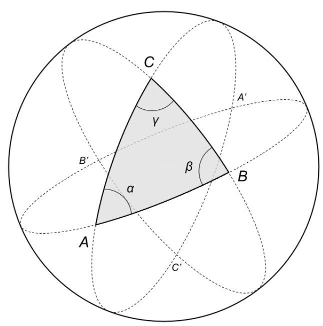

Figure 2: Triangle formed by lunes on a sphere.

Question 2.3. *What is the area of this triangle if the sphere has radius r?*

This question will highlight an example of geometry where area is defined differently than in the Euclidean setting, which will help us understand triangles in hyperbolic geometry better.

Lemma 2.4. *The area of a lune with angle α is*

$$2\alpha r^2. \tag{1}$$

Proof. We know that the the area of a circle is $2\pi r^2$. We can pick the center of the sphere point O depicted in Figure 1 and create a great circle. Let the top and bottom of the sphere perpendicular to the great circles be the two antipodal points of the lune. Let α be the angle of the lune in the circle (in radians). Since 2π is the degree measure of entire circle, $\frac{\alpha}{2\pi}$ is the angle of the lune proportional to the entire circle.

Furthermore, we know that the surface area of a sphere is $4\pi r^2$. Hence, using proportions, the surface area of the lune of angle α is

$$\frac{\alpha}{2\pi} \cdot (4\pi r^2) = 2\alpha r^2.$$

□

Theorem 2.5. *The area of a spherical triangle is proportional to the difference between its angle sum and π. More precisely, on a sphere with radius r, the area of a spherical triangle with angle measures of α, β, and γ is $(\alpha + \beta + \gamma - \pi)r^2$ (see Figure 2).*

Proof. By equation (1), the area of these lunes are

$$\begin{aligned} Area(ABA'C) &= 2\alpha\pi r^2 \\ Area(BAB'C) &= 2\beta\pi r^2 \\ Area(CAC'B) &= 2\gamma\pi r^2. \end{aligned}$$

The surface the three lunes cover is equal to half of the surface area of the sphere because they cover a symmetric region. Furthermore, as the surface area of a sphere is $4\pi r^2$, we have

$$\begin{aligned} 2\pi r^2 &= Area(ABA'C) + Area(BAB'C) + Area(CAC'B) - 2Area(\triangle ABC) \\ 2\pi r^2 &= 2\alpha r^2 + 2\beta r^2 + 2\gamma r^2 - 2Area(\triangle ABC) \end{aligned}$$

using Lemma 2.4. Then,

$$\begin{aligned} 2Area(\triangle ABC) &= 2\alpha r^2 + 2\beta r^2 + 2\gamma r^2 - 2\pi r^2 \\ Area(\triangle ABC) &= (\alpha + \beta + \gamma - \pi)r^2. \end{aligned}$$

□

The main thing to note here is that triangles act very differently than on Euclidean space. Rather than having an area depending solely on the side lengths, we have that the area depends on the angles formed by the great circles. Also note that the area always being non-negative implies that $\alpha + \beta + \gamma \geq \pi$, i.e. the angle sum of a triangle in spherical geometry is more than 180°.

3 Axiomatic Systems

The bedrock of understanding these different framings of geometry are different *axiomatic systems*. They act as the basis for all geometric proofs within geometry, and thus they must be thoroughly proven.

A characteristic difficulty with axiomatic systems is the problem of undefined terms. These are terms that cannot be defined within the system, and so an understanding of them rests on the general intuition of the student and approximate visual or verbal descriptions. With our current understanding of geometry, the only three necessarily undefined terms are *point*, *line*, and *plane*. It is important to note that, while we do provide definitions for these terms, they are still undefined within this systems because their definitions cannot be derived from the other definitions in the axiomatic system.

In order to keep the following sections orderly, we will define our undefined terms for Euclidean geometry as the following:

- A *point* is constituted as a location in space with no length, width, or area
- A *line* is constituted as an indefinitely long, straight length of space with no width or area
- A *plane* is an indefinitely expansive, two-dimensional, flat surface with length and width, but no height

Euclidean geometry, sometimes generalized as geometry that occurs on a flat plane, is what is taught and studied in high school geometry. It is built on the following five postulates (axioms) that form the axiomatic system of Euclidean geometry:

Euclid has the following five main axioms for Euclidean geometry:

1. We can draw a straight line from any point to any other point.
2. We can "continue" a line segment continuously into a straight line.
3. We can construct a circle so that every point along the edge is equidistant from the center.
4. The measure of right angles are always $90°$ and equal.
5. If we have a line intersecting two other lines at two distinct points, and the sum of the measure of the interior angles formed between them is less than π, then the lines will intersect on that side (depicted in Figure 3).

Most of these are fairly intuitive, and can even feel cyclical from where we are with modern geometry; however, all five are integral to our understanding of geometry on this plane. For years, those who studied Euclid's first Five Postulates, they reasoned that, while the first four were intuitive and stood independent from the others, the fifth seemed like it should be a theorem, not a postulate: derived from the first four and unnecessary from the Let us examine the fifth in greater detail. As it can be difficult to visualize, we will use the following diagram to inform the exploration:

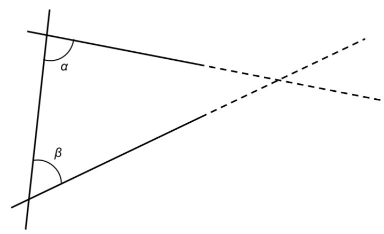

Figure 3: A visual representation of Euclid's Fifth Postulate.

This visualization makes sense to us as we often perceive the world through Euclidean Geometry, at least on a small scale. In Euclidean geometry, there will always be only one line through point P that will not intersect k. However, many years after Euclid's death, people began asking the question:

Question 3.1. *What if there could be more?*

4 Hyperbolic Geometry

Giovanni Girolamo Saccheri, a mathematician of the early seventeenth- and early eighteenth-centuries, was one of these questioners, and was unconvinced that the Fifth Postulate was even necessary. In his efforts to prove the superfluity of Euclid's Fifth Postulate, he assumed the Postulate was false and reached two conclusions that were, in his words: "repugnant to the nature of the straight line." One of his conclusions stated that there is a line through point P would intersect k not once, but *twice.* The other stated that there existed two lines through P that would not intersect k at all. These, respectively, are the characteristic axioms of spherical geometry and, particularly of interest, hyperbolic geometry (Figure 5).

Axiom 4.1 (The Characteristic Axiom of Hyperbolic Geometry)**.** *Given a line k and a point P not on k, there exists at least two lines m and l that do not intersect k.*

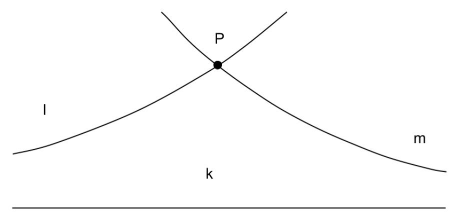

Figure 4: The Characteristic Axiom of Hyperbolic Geometry (Axiom 4.1)

4.1 Pasch's Axiom and Infinitely Many Lines

To gain an understanding of hyperbolic geometry, we must first begin with an axiomatic system. The undefined terms relied on are *point, line, on, between,* and *congruent*, and the axioms are the same as those above for Euclidean geometry, however replacing the Fifth postulate with Saccheri's derived contradiction.

From these assumptions, we can start to piece together the properties of hyperbolic geometry. One way to do this is by carrying over theorems and axioms from Euclidean geometry and seeing how they change. For instance: one interpretation of the Fifth Postulate, Pasch's Axiom, translates well into hyperbolic geometry and becomes the basis for a majority of derived hyperbolic geometry.

Axiom 4.2 (Pasch's Axiom)**.** *If a line enters a triangle through one side, it will exit through either one of the opposite two sides or the opposite point.*

Theorem 4.3. *Given a line k and a point P not on k, there are infinitely many lines that do not intersect k.*

In Euclidean space, the axiom can be simplified to assumption that: if a line enters a triangle through one side, it will not leave the triangle through the same side. With our revised axioms, this carries over to hyperbolic space and from it, we can construct the following proof:

Proof. Lines l and m are the two parallel lines to k, meeting at point P. Choosing a point on k, A, and a point on l, B, construct a triangle $\triangle ABC$. Line m enters $\triangle ABC$ through point P and exits through point D, between A and B. Defining point X as an arbitrary point on $\overline{DB}$, meaning it could be any one of infinitely many we can draw $\overleftrightarrow{PX}$.

Now, working under this derivation, we can conclude that $\overleftrightarrow{PX}$ does not intersect line k because, if it did, for instance at the hypothetical point Y, line m would violate Pasch's Axiom. Entering triangle $\triangle XYA$ at point D, it would

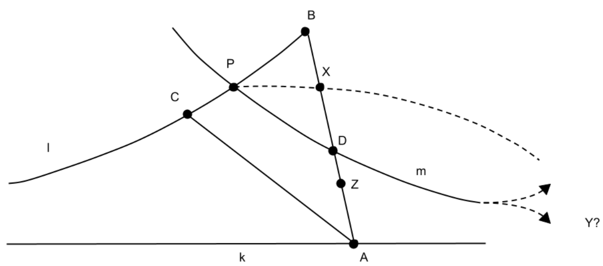

Figure 5: Pasch's Axiom in hyperbolic space.

have to exit through $\overline{XY}$ or $\overline{YA}$, but it already intersects $\overleftrightarrow{PX}$ (of which $\overline{XY}$ is a part of) at P and— by definition— cannot intersect $\overline{YA}$ because $\overline{YA}$ is part of line k, to which it is parallel. Therefore, we can work under the assumption that there are infinitely many lines parallel to k that go through point P because of the infinitely many points X between B and D. $\square$

4.2 Sensed parallels

As proven in Theorem 4.3, there are infinitely many lines on P that have no points on k. In this way, there are infinitely many lines 'parallel' to line k that pass through point P. However, this does not mean that every single parallel line through point P is the same distance away from line k. In 2D Euclidean space, we can imagine two parallel lines. As one begins tilting by a small degree, it will eventually intersect at a point far away from the center of its rotation. As the amount of rotation increases, the distance from the center of rotation to the intersection of lines becomes closer, summarizing Hilbert's axiomatic approach to geometry and his parallel postulate.

Axiom 4.4. *In the Euclidean plane there can be drawn through any point P, lying outside of a straight line k, one and only one straight line which does not intersect the line k. This straight line is called the parallel to k through the given point P.*

In hyperbolic space, we can develop a similar picture. Using Axiom 4.1, we can see that as a parallel line to k becomes closer and closer to line k, there is a tipping point where the line cannot tilt any further without intersecting. This line is called the *sensed parallel.*

Definition 4.5. Given point P not on line k, the first line on P in each direction that does not intersect k is the (right- or left-) sensed parallel to k at P. Other lines on P that do not intersect k are called *ultraparallel* to k.

4.3 Angles of Parallelism

The angles of parallelism are used in hyperbolic geometry to determine the angle of a sensed or an ultra parallel in hyperbolic geometry as it approaches line k. This concept cannot be applied in Euclidean geometry since the angle of parallelism is constant. In other words:

Definition 4.6. By constructing line segment $\overleftrightarrow{AP} \perp$ to k, in Axiom 4.1, the acute angle formed by $\overleftrightarrow{AP}$ with l and m are the **angles of parallelism** at P

Theorem 4.7. *If l and m are the two sensed parallels to k at P, they have the same angle of parallelism.*

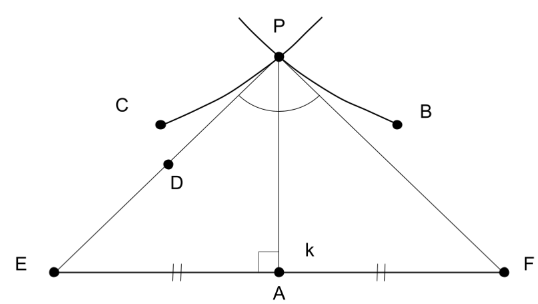

Figure 6: Angles of parallelism.

Proof. This proof is similar to the characteristic axiom in the sense that sensed parallels approach line k without intersecting. We let $\overline{\mathrm{AP}} \perp k$, and lines $\overline{\mathrm{PB}}$ and $\overline{\mathrm{PC}}$ be the sensed parallel to k through P. Proving through contradiction, let angles of parallelism differ, let, $\angle$ APB $<$ $\angle$ APC. Inside $\angle$ APC, construct $\angle$ APD $\cong$ $\angle$ APB. Because line PC is a sensed parallel, line PD intersects k, say at E, constructing $\triangle APE$. Let F be on k with $\overline{\mathrm{AF}} \cong \overline{\mathrm{AE}}$. By SAS, $\triangle$ APF $\cong$ $\triangle$ APE. But then $\angle$ APF $\cong$ $\angle$ APD $\cong \triangle APB$, meaning lines PF and PB are the same by Hilbert's axiom III-4. However, line $\overline{\mathrm{PB}}$ is a sensed parallel to k, so it cannot intersect k at F, which is a contradiction so their angles of parallelism must be the same. $\square$

From this proof we come to a conclusion vital to the future proofs presented in the paper.

Corollary 4.8. *All angles of parallelism are less than a right angle. Two lines with a common perpendicular are ultraparallel.*

Proof. The angle of parallelism is the angle a sensed parallel makes with the perpendicular line so therefore the angle has to be less than $90°$. If the angle was equal to $90°$, the two sensed parallels would be the same, contradicting the characteristic axiom (Axiom 4.1). $\square$

Since hyperbolic space is hard to picture visually, we use models to depict them more easily. For example, the Poincaré model uses a circular boundary as a plane and semi-circular lines in Euclidean space as straight lines in hyperbolic.

4.4 Omega Triangles

When sensed parallels are introduced in their asymptotic nature, it prompts the question:

Question 4.9. *Could one construct a triangle using sensed parallels as two sides?*

If we consider sensed parallels as "meeting at infinity" rather than getting infinitely close, then we deduce the same definition that the framers of hyperbolic geometry came to: the *omega point*, an imaginary point at the infinite limit of the hyperbolic plane. Using one omega point two rays leading to it with corresponding points A and B, we can form an omega triangle $\triangle AB\Omega$, pictured below using the Poincaré model of the hyperbolic plane:

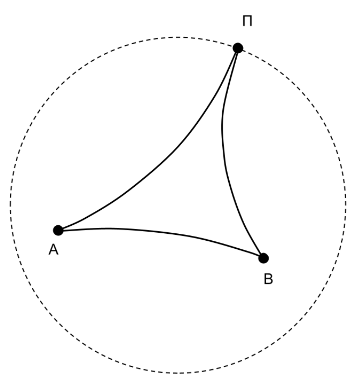

Figure 7: An omega triangle in the Poincaré model for hyperbolic space, the omega point here represented by Π.

As with parallels, some postulates carry over from Euclidean geometry. However, there are a couple more interesting theorems worth exploring such as the following:

Theorem 4.10. *The measure of an exterior angle of an omega triangle is greater than the measure of the opposite interior angle.*

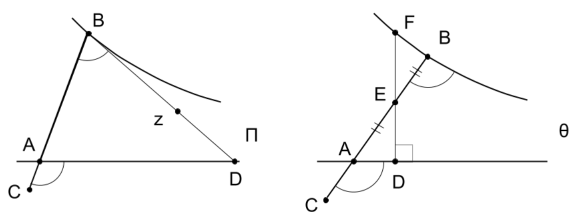

Figure 8: Exterior angles of omega triangles.

Proof. Consider the construction in figure 8. We prove $\angle CA\Pi > \angle AB\Pi$ through contradiction.

Suppose for the sake of contradiction that $\angle CA\Pi < \angle AB\Pi$. We form $\angle ABZ$ inside of $\angle AB\Pi$ such that $\angle ABZ \cong \angle CA\Pi$. Then, we say $\overleftrightarrow{BZ}$ intersects $\overrightarrow{A\Pi}$ at point D because $\overrightarrow{B\Pi}$ is sensed parallel to $\overrightarrow{A\Pi}$ at B. However, then $\angle CA\Pi$ is an exterior angle to to the ordinary triangle $\triangle ABD$ with the opposite interior angle of $\angle ABD$, which is congruent to $\angle CA\Pi$. This contradicts Euclid's Sixteenth Postulate, which dictates that the exterior angle of a triangle is greater than either remote interior angle.

Furthermore, suppose for the sake of contradiction that $\angle CA\Theta \cong \angle AB\Theta$. Let E be the midpoint of $\overline{AB}$ and construct $\overline{DE}$ such it lies perpendicular to $\overleftrightarrow{A\Theta}$. As mentioned in the above section, the angle of parallelism is acute so we can assume D is not A. If we construct F on $\overleftrightarrow{B\Theta}$ such that $\overline{FB} \cong \overline{AD}$ and F and D are on opposite sides of $\overleftrightarrow{AB}$, as shown in Figure 8, we construct two congruent triangles by SAS: $\triangle FBE$ and $\triangle DAE$, assuming $\angle FBE \cong \angle DAE$. From this we derive that, because $\angle ADE$ is a right angle, $\angle BFE$ must be as well. However, this presents the contradiction that the angle of parallelism for $\overleftrightarrow{F\Theta}$ to $\overleftrightarrow{D\Theta}$ must be right, which contradicts the corollary that the angle of parallelism must be less than a right angle. Since both of the other options are contradictions, the theorem is proved. $\square$

From here, we can begin an exploration of congruency in omega triangles. Because the sides extending to infinity are infinitely long and the angle where they meet is infinitely small, the only points of interest are the two angles and the side between them.

Definition 4.11. Two omega triangles, $\triangle AB\Omega$ and $\triangle CD\Gamma$, are congruent if and only if $\overline{AB} \cong \overline{CD}$, $\angle BA\Omega \cong \angle DC\Gamma$, and $\angle AB\Omega \cong \angle CD\Gamma$.

4.5 Saccheri Quadrilaterals

One of Saccheri's explorations in hyperbolic space led to the definition of Saccheri quadrilaterals. Discovered centuries earlier by the Persian mathematician Omar Khayyam (albeit not named after him), the definition is as follows:

Definition 4.12. A **Saccheri quadrilateral** has two opposite congruent sides perpendicular to one of the two other sides. The perpendicular is the *base* and the fourth side is the *summit*.

Saccheri quadrilaterals can exist in hyperbolic and spherical space, with concave and convex summits respectively, however we will focus only on the hyperbolic instance in this paper. Consider the following theorems:

Theorem 4.13. *The summit angles of a Saccheri quadrilateral are congruent. The base and summit are perpendicular to the line on their midpoints.*

Proof. Let $\overline{AB}$ be the base of Saccheri quadrilateral $ABCD$ and $\overline{AC}$ and $\overline{BD}$ be the diagonals, as pictured in the first diagram in Figure 10. Thus it follows that $\triangle ABC \cong \triangle BAD$ by SAS and that $\triangle ADC \cong \triangle BCD$. From this we can conclude that summit angles $\angle ADC \cong \angle BCD$. Naming points E and F as the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. We can follow a similar line of logic stating that $\triangle DAE \cong \triangle CBE$ by SAS and that $\triangle DEF \cong \triangle CEF$ by SSS. This means that $\angle DEF \cong \angle CEF$ and $\overline{EF} \perp \overline{CD}$ and $\overline{AB}$ because the angles form a straight angle. $\square$

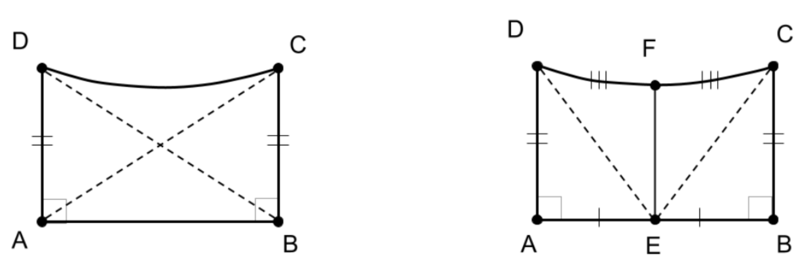

Figure 9: Saccheri quadrilaterals.

Theorem 4.14. *The summit angles of a Saccheri quadrilateral are acute.*

Proof. Sensed parallels to lines $\overleftrightarrow{AB}$, $\overleftrightarrow{C\Pi}$ and $\overleftrightarrow{D\Pi}$, intersect Saccheri quadrilateral $ABCD$ with base $\overline{AB}$ at C and D, forming $\angle ED\Pi$ with point E on line $\overleftrightarrow{CD}$. The angle $\angle ED\Pi$ is the exterior angle of omega triangle $\triangle DC\Pi$, making it larger than $\angle DC\Pi$. Because $\overline{AD} \cong \overline{BC}$ and $\angle \Pi AD \cong \angle \Pi BC$, we can infer that the corresponding omega triangles are congruent, meaning $\angle AD\Pi \cong \angle BC\Pi$ because they are the angles of parallelism. Thus, $\angle EDA > \angle DCB$, and, since the summit angles are equal, $\angle EDA$ is also bigger than $\angle CDA$. The smaller is acute because line $\overleftrightarrow{CD}$ is straight. □

4.6 Angle sum of a Triangle

In Euclidean space, it has been proven that the sum of the interior angles of a triangle always adds up to π. In hyperbolic space, the sum of the interior angles of a triangle is less than π.

Theorem 4.15. *The angle sum of a triangle is less than π.*

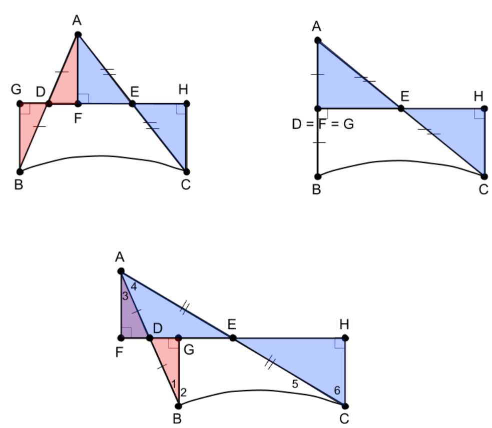

Figure 10: The angle sum of a triangle is less than π.

Proof. In $\triangle ABC$, let D be midpoint of $\overline{\mathrm{AB}}$ and E be midpoint of $\overline{\mathrm{AC}}$. Construct perpendiculars $\overline{\mathrm{AF}}, \overline{\mathrm{BG}}, \overline{CH}, \overline{\mathrm{DE}}$. We claim three cases of the geometric configuration of $\triangle ABC$.

Case 1) $\angle ABC < \angle GBC$. Case 1 corresponds to Figure 4.25. We know that $\overline{AD} \cong \overline{BD}$, $\angle AFD = \angle BGD = 90°$, and $\angle ADF = \angle BDG$. Hence by AAS, $\triangle ADF \cong \triangle BDG$. Similarly, by AAS $\triangle AEF \cong \triangle CEH$. The right angles at G and H with the congruent sides show that $GHCB$ is a

Saccheri quadrilateral. Because of Theorem 4.12, the summit angles of a Saccheri triangle are acute or $< 180°$. As follows from AAS, $\angle DAF = \angle GBD$ and $\angle EAF = \angle ECH$.

Hence, $\angle DAF + \angle DBC = \angle GBC$ and $\angle EAF + \angle ECH = \angle HCB$. Therefore, $\angle GBC + \angle HCB < 180°$.

Case 2) $\angle ABC = \angle GBC$. Case 2 corresponds to Figure 4.26. Similar to Case 1, since $\triangle AED \cong \triangle EHC$, $\angle EAD = \angle ECH$. Since $\angle ABC$ is already a summit angle and $\angle EAD$ is congruent to 1, $\triangle$ is less than 180°.

Case 3) $\angle ABC > \angle GBC$. Case 3 corresponds to Figure 4.27. By AAS, $\triangle AFD \cong \triangle BGD$. To prove triangle has angle sum less than π we need to prove it is equal to summit angles of triangle. Following Figure 4.27, we have

$$\begin{aligned}
\angle ABC + \angle BAC + \angle ACB &= \angle 2 + \angle 1 + \angle 4 + \angle 5 \\
&= \angle 2 + \angle 3 + \angle 4 + \angle 5 \\
&= \angle 2 + \angle 6 + \angle 5 \\
&= \angle GBC + \angle HCB.
\end{aligned}$$

Therefore, in all three cases, we have shown that the angle sum of a triangle in hyperbolic geometry are less than π. □

We have only scratched the surface of hyperbolic geometry. Just like Euclidean geometry, there are multiple distance and area formulas that we will not discuss in this paper. When we removed Euclid's Fifth postulate, we were able to define the characteristic axiom of hyperbolic geometry. In a similar way, we can define the characteristic axiom for spherical geometry, making it a bit more rigorous.

5 Single Elliptic Geometry

In the 1800s, spherical geometry was just thought of as a subsection of Euclidean geometry (after all, a circle is a portion of the plane, and the sphere is a portion of 3D Euclidean space). However, they are *very* different (as we have seen so far by the area of a triangle in spherical geometry. We may yet again replace Euclid's fifth postulate with a new characteristic axiom.

Axiom 5.1 (The Characteristic Axiom of Spherical Geometry)**.** *Two lines (interpreted as great circles) always intersect in two points*

In 1874, Felix Klein created a modified spherical geometry, called single elliptic geometry, to obey the similar principles of Euclidean and spherical spaces where two points determine a line. One common model describes a "borderless" hemispherical geometry, where lines continue from the point opposite the point on the edge that they exited; "exiting" through one edge and then continuing from the antipodal point to where it "exited", yet technically unbroken. Klein referred to this as single elliptic geometry. Thus, the characteristic axiom for single elliptic geometry states that:

Axiom 5.2 (The Characteristic Axiom of Single Elliptic Geometry)**.** *Two distinct lines intersect in exactly one point.*

This is because the hemisphere has no true great circles– the closest being the edge of the space which is discounted because it doesn't exist, it just "teleports" the continuation of the line to the other side. As such, no line can exist on the edge. Therefore the only lines that can exist are ones with an angle to the edge that does not equal 0 or π, and those can intersect only at one point.

Although spherical and single elliptic are different geometries, they share many theorems in common. For instance, the angle sum of a triangle is greater than π, and triangles can have three obtuse angles (note that this implies that the maximum angle sum is $< \pi$.

We can also produce Saccheri quadrilaterals in spherical geometry. We accept that Theorem 4.7 holds in spherical space and propose an integral new theorem for spherical geometry:

Theorem 5.3. *In single elliptic geometry, all lines perpendicular to a given line intersect in one point.*

Proof. Constructing spherical Saccheri quadrilateral $ABCD$, we accept that $\overline{EF} \perp \overline{AB}$ and $\overline{CD}$. Because of the characteristic axiom of single elliptic geometry, we say that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect in two opposing points: P and P', and that $d(A,P) = d(B,P)$ and $d(C,P) = d(D,P)$. To begin a contradiction, we say that $\triangle ADP \cong \triangle BCP'$ by SAS. It follows that $\angle BCP' \cong \angle ADP$ and is supplementary to $\angle BCD$. With Euclid's Sixteenth Postulate, we can then say $\overleftrightarrow{CP'}$ is $\overleftrightarrow{CD}$, which results in the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. This means that point P is point P', and thus the assumption about distance holds. We can also infer that E and F represent the maximum distance two points can be separated, and thus form an isosceles triangle: $\triangle EFP$ with $d(E,P) = d(F,P)$ by Euclid's Sixth Postulate. □

We can also come to the following conclusion about spherical Saccheri quadrilaterals:

Theorem 5.4. *In single elliptic geometry, the summit angles of a Saccheri quadrilateral are obtuse.*

Proof. Again constructing Saccheri quadrilateral $ABCD$, we construct $\overleftrightarrow{DG}$ such that it lies perpendicular to $\overline{AD}$. Point Q is the intersection of $\overleftrightarrow{DG}$ with $\overleftrightarrow{AB}$. From this, we can posit that $d(A,P) \leq d(A,Q)$, because point Q is the farthest point from A, as proven in Theorem 5.2. If we were to accept that the distances were equal, we would construct a trivial quadrilateral because P would equal Q and, following from this, $A = E$ and $A = B$. So $d(A,P) < d(A,Q)$ and $\overleftrightarrow{DP}$ enters $\triangle QDA$ at D, so $\angle PDA < \angle QDA$. Since $\angle QDA$ is right, the summit angle $\angle ADC$— the supplementary angle— must be obtuse. □

6 Summary and Conclusion

Geometry is not set in stone and was not completely understood by Euclid two-thousand years ago. It is a constantly changing field that, for all we know, could be totally different in the next two-thousand years. Contemporary high school geometry spends less time focusing on developing axiomatic systems since proofs must directly and carefully follow the axioms.

One mathematician, David Hilbert, avoided writing a text with all of the familiar and proven theorems but instead wanted to assume as little as possible while proving all Euclidean theorems. However, the consequences of this idea were long and rigorous proofs that were hard to follow, making it incompatible as an introduction to geometry. It took Hilbert eleven axioms to define the properties of a line, something that the high school axiomatic system does in four. While we want to assume as little as possible in order to keep theorems clear and concise, there is a tipping point as to how many axioms are necessary for non-rigorous and straightforward proofs.

Through rigorous axiomatic systems, we can define various geometries by characteristic axioms. The theorems that follow present an interesting mixture between Euclid's constructive and Hilbert's rigorous methodology.

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References

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